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Synthesis of Mixed Lumped and Distributed Impedance-Transforming Filters

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***Abstract*—The design of a class of impedance-transforming filters in the form of very compact and convenient mixed lumped and distributed ladder networks is presented. The synthesis utilizes the distributed prototype technique introduced in a previous paper, but here a new approximation function appropriate to the impedance transformer problem is derived. In addition to combining the properties of an impedance transformer and a low-pass filter, the new circuit represents a solution to the problem of short-line matching to an extreme impedance value without using extreme impedance values in the transformer. Broad-band designs are tabulated for a wide range of parameters. A discussion of the application of the technique in the design of mixed lumped and distributed broad-band matching networks is included.**

A 50–10-Ω transformer was designed for the band 3.5–7.0 GHz, having a voltage standing-wave ratio of 1.15 and giving an attenuation >20 dB in the band 10.5–21.0 GHz. The length of this transformer is 0.875 in, and the experimental results showed excellent agreement with theory.

Introduction

THE design of impedance transformers in a compact and convenient format has been the subject of considerable research effort. The conventional (and usually best) solution to the problem of matching between two purely resistive impedances is the well-known multiquarter-wave section stepped impedance transformer. However, this can be rather lengthy, especially at the lower microwave frequencies. The short-step impedance transformers described by Matthaei *et al.* [1] are, as the name implies, much shorter than the corresponding quarter-wave transformers. Their chief disadvantage is the fact that the range of impedance levels within any given transformer is considerably larger than the input and output impedance levels. For example, a six-section $\lambda_m/16$ transformer for a 5:1 impedance change and fractional bandwidth $\omega=0.8$ requires normalized impedances varying between 0.572 and 8.74, a range of 15.25:1, compared with only 5:1 for the transformer ratio obtained. Thus a transformer from 10 Ω to 50 Ω utilizing this design would require one line having the very low impedance value of 5.72 Ω.

This example illustrates the problem of producing short-length impedance transformers where the impedance levels within the transformer lie in a range encompassed by the terminating impedances. Preferably it would be most desirable to design a short transformer to an extreme terminating impedance without requiring such an impedance in the transformer. Conventional quarter-wave transformers achieve this at the expense of length.

In one sense a solution to the problem can be obtained with mixed lumped and distributed circuits. A simple way to see this is to note that we might consider replacing the $\lambda_m/16$ line of impedance 5.72 Ω in the Matthaei transformer described by a lumped capacitor. This can be accomplished rather accurately since the line is mainly capacitive. If all the low-impedance lines

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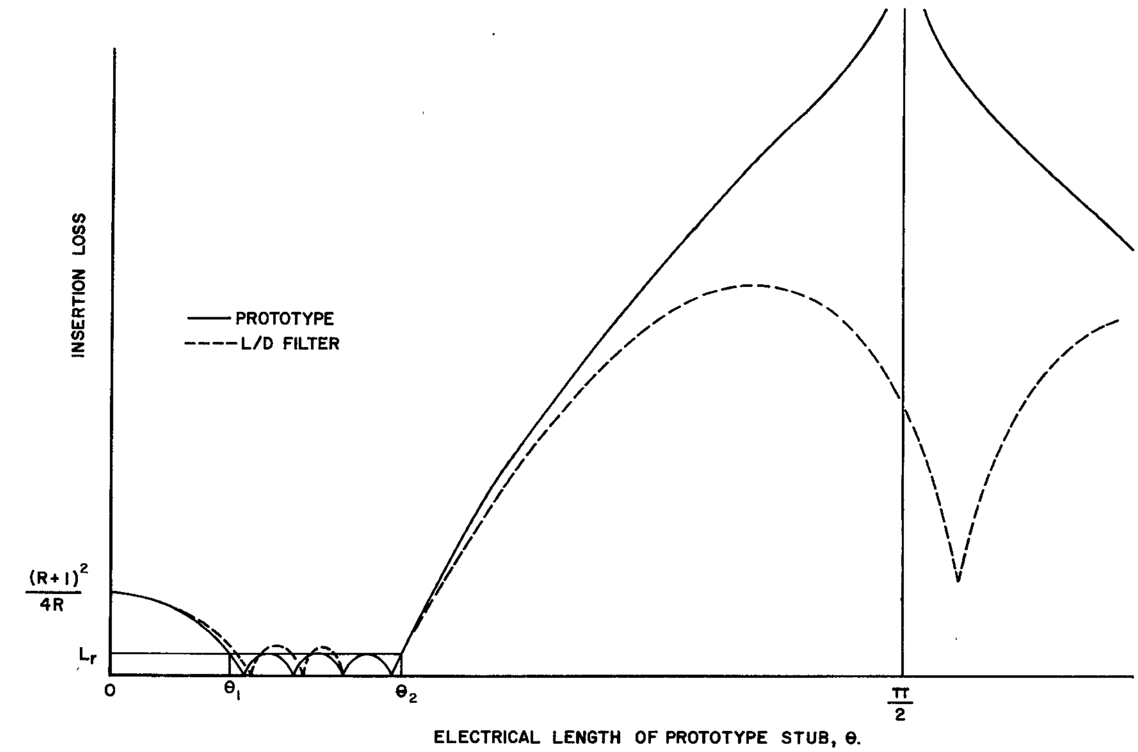

Fig. 1. Comparison between insertion loss of distributed prototype and mixed lumped and distributed impedance-transforming filters.

in the transformer could be converted into capacitors, the desired result would be obtained partially, while the problem of the high-impedance lines remains.

The problem of realizing the low-impedance lines reduces to that of realizing lumped or substantially lumped capacitors at microwave frequencies. One way to do this is to use very short lines of low impedance, which may appear to present us with the same problem as before. This is in fact not quite the case, since the construction of microwave disk capacitors, which are commonly used in coaxial low-pass filters, is not difficult. The transformer becomes even shorter by the compression of the low-impedance lines into lumped capacitors. In turn, this implies that we could have commenced from a somewhat longer prototype having a lesser range of impedance levels. However Matthaei notes [2] that the basic line length must be less than $\lambda_m/8$. This restriction is not fundamental to mixed lumped and distributed transformers, which when designed by the method described herein can give very convenient impedance levels.

It should be noted also that there has been considerable interest in using lumped components in microwave integrated circuits [3]–[6]. Microwave chip capacitors are quite common, microwave lumped inductors rather less common but in existence. The utilization of mixed lumped and distributed circuits would appear to offer additional advantages and flexibility of design in integrated circuits.

A further advantage of mixed lumped and distributed transformers is the possible allowance for parasitics in the terminating impedances, i.e., these need not be purely resistive. The type of impedance-transforming filter described in this paper allows the absorption of one or two parasitic elements across a terminating resistor, e.g., a series inductance and a shunt capacitance, but the principle may be extended to more complicated terminations.

Finally one of the most useful aspects of the new circuits is their harmonic filtering property. A conventional quarter-wave transformer gives little or no attenuation at the higher harmonics. On the other hand the class of mixed lumped and distributed circuits described here combines the properties of an impedance transformer and a low-pass filter. It should be useful in, for example, the design of varactor multipliers or any application where combined filtering and impedance transformation is desirable.

Another type of impedance transforming filter having an interdigital realization is described by Wenzel [7]. This would normally be more complicated and expensive than the quasi-low-pass filter described here.

The Method

The synthesis technique for the mixed lumped and distributed impedance-transforming filters is based on a class of fully distributed prototype filters consisting of a cascade of open-circuited shunt stubs of electrical length θ spaced by transmission line elements (TLE)[1] of

[1] It is suggested that the misleading term "unit element" commonly used for a length of transmission line should be replaced by the more direct and descriptive term "transmission line element."

length 2θ [8]. Since the stubs are electrically short for θ less than approximately $\pi/4$, they may be replaced by lumped capacitors, and the filter will remain well matched in a low-pass band. For example, a stub of admittance y_i and electrical length θ_2 at frequency ω_2 may be replaced by a capacitor C_i given by the equation

$$\omega_2 C_i = y_i \tan \theta_2. \quad (1)$$

In a low-pass filter this equivalence is arranged to be satisfied at the cutoff frequency, which is therefore realized exactly. The procedure is the converse of designs which commence from a lumped element low-pass prototype. Once the prototype is available, it is also considerably simpler to implement.

Examples presented in [8] and further examples given later in this paper show the extent of the deviations between the mixed lumped and distributed filter and the prototype on which it is based. The deviations are due to the differences between the admittances $y_i \tan \theta$ and ωC_i shunted across the main line by the ith stub of the prototype and the corresponding capacitor of the mixed lumped and distributed filter. Below the frequency ω_2 at which (1) is satisfied, $\omega C_i > y_i \tan \theta$, but the reflection coefficients at the junctions vary in the same way with frequency. The passband ripples increase in amplitude slightly as ω decreases. Above the frequency ω_2, $\omega C_i < y_i \tan \theta$, and the attenuation of the mixed lumped and distributed filter is less than that of the prototype. The prototype has a pole of attenuation when θ is an odd multiple of $\pi/2$, whereas the mixed lumped and distributed filter reaches a finite maximum of attenuation, as illustrated in Fig. 1.

In the case of an impedance-transforming quasi-low-pass filter, there is a mismatch at zero frequency whose voltage standing-wave ratio (VSWR) is given by the ratio R of the terminating resistances. It is necessary to seek such a prototype insertion loss function which is equiripple between frequencies proportional to electrical lengths θ_1 and θ_2, as shown in Fig. 1. Additionally the prototype must consist of a cascade of open-circuited shunt stubs of electrical length θ spaced by TLEs of length 2θ. A minimum number of TLEs of length θ may also be tolerated.

The lowest ordered prototype consists of a single TLE with stubs at each end, as shown in Fig. 2. The transfer matrix of this circuit is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \frac{1}{1-t^2}\begin{bmatrix} 1 & 0 \\ y_1 t & 1 \end{bmatrix}\begin{bmatrix} 1+t^2 & 2t/Y_1 \\ 2Y_1 t & 1+t^2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ y_2 t & 1 \end{bmatrix}$$
$$= \frac{1}{1-t^2}\begin{bmatrix} 1+(1+2y_2/Y_1)t^2 & 2t/Y_1 \\ (y_1+y_2+2Y_1)t+(y_1+y_2+2y_1y_2/Y_1)t^3 & 1+(1+2y_1/Y_1)t^2 \end{bmatrix} \quad (2)$$

where t is Richards' variable, i.e., at real frequencies $t = j \tan \theta$. When terminated in impedances 1 and R, the insertion loss of the lossless reactive network is

$$\frac{P_0}{P_L} = 1 + \tfrac{1}{4}\left| A\sqrt{R} - D/\sqrt{R} \right|^2 + \tfrac{1}{4}\left| B/\sqrt{R} - C\sqrt{R} \right|^2. \quad (3)$$

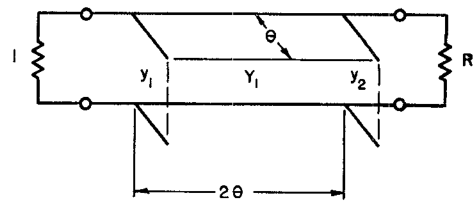

Fig. 2. Lowest ordered prototype.

It would be very difficult to discover general insertion loss functions of this type containing two complicated squared terms, the sum of which gives an equiripple response. In practice it is necessary to restrict the insertion loss function to be either symmetric or antimetric. The symmetric case does not apply to an impedance transformer because of the necessity of having a mismatch at $t=0$, so that we seek a solution for the antimetric case where

$$B/\sqrt{R} = C\sqrt{R}. \quad (4)$$

It is immediately apparent that a solution of this antimetric type cannot be obtained from (2), since the B and C terms are not of equal degree. The general case of n TLEs of length 2θ has a transfer matrix of degree

$$D\begin{bmatrix} 2n & 2n-1 \\ 2n+1 & 2n \end{bmatrix} \quad (5)$$

where D denotes degree.

In order to make the B and C terms equal in degree it is necessary either to increase the cascade by a single TLE of length θ, or to use a prototype having a transmission zero of order 2. In the case of the extra TLE, the transfer matrix of which is

$$\frac{1}{\sqrt{1-t^2}}\begin{bmatrix} 1 & t/Y \\ Yt & 1 \end{bmatrix} \quad (6)$$

the degree of the overall network becomes

$$D\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot D\begin{bmatrix} 2n & 2n-1 \\ 2n+1 & 2n \end{bmatrix} = D\begin{bmatrix} 2n+2 & 2n+1 \\ 2n+1 & 2n \end{bmatrix}. \quad (7)$$

Matrix (7) is now capable of satisfying condition (4). However, the network is terminated with a single-length TLE having no shunt stub at one end adjacent to a resistive termination. This configuration is not very useful if it is desired to match out the parasitic shunt capacitance which the termination might possess at that end of the transformer. It is useful in some cases where such a parasitic exists at only one end, but even then

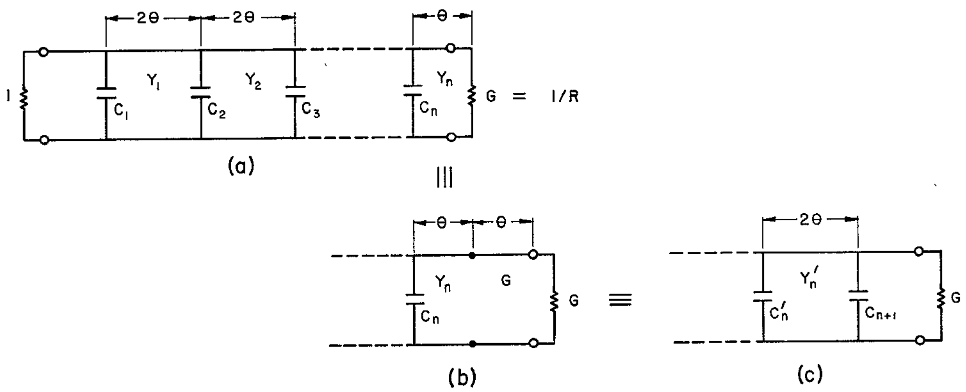

Fig. 3. (a) Basic single-ordered prototype with TLE at one end. (b) Addition of redundant TLE. (c) Final network. Lumped capacitors shown indicate stubs of length θ in prototype. $C_n - C_n' = (G - Y_n)/(1 + G/Y_n)$, $Y_n' = 2G/(1 + G/Y_n)$, $C_{n+1} = G(G - Y_n)/(G + Y_n)$.

TABLE I

SINGLE-ORDERED TRANSMISSION ZERO, $n = 2$, $\theta_1 = 15°$, $\theta_2 = 30°$

R	2	3	4	5	6	8	10
STUB							
1	1.4487	1.9029	2.2168	2.4656	2.6757	3.0256	3.3171
2	2.5870	3.7407	4.7071	5.5708	6.3639	7.8007	9.0943
3	.7908	1.4466	2.1243	2.8224	3.5374	5.0073	6.5189
LINE							
1	.8592	.9594	1.0483	1.1268	1.1972	1.3208	1.4282
2	1.2092	1.5534	1.8757	2.1776	2.4626	2.9927	3.4811
VSWR	1.1634	1.2800	1.3772	1.4638	1.5434	1.6893	1.8236
THETA	ATTENUATION OF L/D FILTER (DB)						
45	6.8442	10.3446	12.3727	13.7783	14.8481	16.4292	17.5882
60	14.5667	18.3154	20.3642	21.7604	22.8154	24.3681	25.5036
75	16.1571	19.4345	21.2452	22.4916	23.4410	24.8511	25.8921
90	9.9390	11.4303	12.2822	12.8973	13.3872	14.1554	14.7568

TABLE II

SINGLE-ORDERED TRANSMISSION ZERO, $n = 3$, $\theta_1 = 15°$, $\theta_2 = 30°$

R	2	3	4	5	6	8	10
STUB							
1	1.1335	1.4026	1.5763	1.7078	1.8149	1.9862	2.1228
2	2.6520	3.3980	3.9749	4.4644	4.8974	5.6512	6.3045
3	2.7835	4.2043	5.5000	6.7230	7.8934	10.1163	12.2190
4	.6313	1.1473	1.6838	2.2386	2.8086	3.9857	5.2019
LINE							
1	.7890	.8359	.8775	.9135	.9453	.9996	1.0454
2	.9177	1.1206	1.3002	1.4620	1.6101	1.8764	2.1134
3	1.3687	1.8527	2.3162	2.7614	3.1914	4.0143	4.7981
VSWR	1.0499	1.0827	1.1087	1.1309	1.1507	1.1855	1.2160
THETA	ATTENUATION OF L/D FILTER (DB)						
45	12.3687	16.3090	18.4604	19.9212	21.0212	22.6335	23.8077
60	24.2039	28.0159	30.0722	31.4666	32.5172	34.0597	35.1855
75	25.6520	28.9146	30.6999	31.9231	32.8523	34.2285	35.2421
90	13.1315	14.3296	15.0474	15.5814	16.0149	16.7057	17.2536

TABLE III

SINGLE-ORDERED TRANSMISSION ZERO, $n = 3$, $\theta_1 = 15°$, $\theta_2 = 45°$

R	2	3	4	5	6	8	10
STUB							
1	.6081	.7913	.9135	1.0085	1.0878	1.2185	1.3265
2	1.1889	1.5757	1.8697	2.1174	2.3356	2.7145	3.0421
3	1.2444	1.8934	2.4598	2.9774	3.4597	4.3453	5.1511
4	.4732	.9307	1.4221	1.9411	2.4825	3.6194	4.8137
LINE							
1	1.0117	1.1083	1.1899	1.2606	1.3238	1.4340	1.5298
2	1.1824	1.4510	1.6858	1.8961	2.0879	2.4316	2.7366
3	1.5268	2.0693	2.5779	3.0589	3.5175	4.3806	5.1863
VSWR	1.1683	1.2887	1.3894	1.4792	1.5619	1.7135	1.8534
THETA	ATTENUATION OF L/D FILTER (DB)						
60	7.8714	11.2936	13.2498	14.6036	15.6346	17.1615	18.2839
75	13.5443	16.7279	18.4982	19.7213	20.6556	22.0476	23.0786
90	8.6336	10.1350	11.0271	11.6855	12.2168	13.0588	13.7234

TABLE IV

SINGLE-ORDERED TRANSMISSION ZERO, $n = 4$, $\theta_1 = 15°$, $\theta_2 = 30°$

R	2	3	4	5	6	8	10
STUB							
1	.8941	1.0722	1.1823	1.2632	1.3278	1.4283	1.5063
2	2.3559	2.8192	3.1515	3.4203	3.6501	4.0358	4.3580
3	3.1621	4.2435	5.1618	5.9853	6.7427	8.1174	9.3585
4	2.6915	4.2099	5.6584	7.0681	8.4495	11.1470	13.7754
5	.4970	.9007	1.3217	1.7580	2.2070	3.1359	4.0980
LINE							
1	.7673	.7860	.8046	.8212	.8359	.8611	.8823
2	.8026	.9242	1.0265	1.1152	1.1941	1.3314	1.4495
3	1.0189	1.3247	1.6056	1.8668	2.1126	2.5689	2.9896
4	1.5030	2.0993	2.6783	3.2420	3.7930	4.8641	5.9020
VSWR	1.0156	1.0256	1.0334	1.0400	1.0457	1.0557	1.0643
THETA	ATTENUATION OF L/D FILTER (DB)						
45	18.3510	22.4228	24.6102	26.0872	27.1962	28.8180	29.9968
60	33.8868	37.7046	39.7576	41.1473	42.1934	43.7276	44.8463
75	35.1488	38.3913	40.1580	41.3654	42.2811	43.6350	44.6307
90	15.5252	16.5507	17.1980	17.6941	18.1043	18.7690	19.3030

TABLE V

SINGLE-ORDERED TRANSMISSION ZERO, $n = 4$, $\theta_1 = 15°$, $\theta_2 = 45°$

R	2	3	4	5	6	8	10
STUB							
1	.4756	.5990	.6778	.7370	.7853	.8625	.9244
2	1.0782	1.3397	1.5265	1.6781	1.8082	2.0279	2.2132
3	1.4157	1.9351	2.3647	2.7444	3.0899	3.7098	4.2625
4	1.2313	1.9553	2.6286	3.2716	3.8920	5.0788	6.2080
5	.3582	.7029	1.0747	1.4687	1.8811	2.7506	3.6684
LINE							
1	.9608	1.0130	1.0570	1.0948	1.1281	1.1855	1.2344
2	1.0617	1.2386	1.3873	1.5169	1.6328	1.8360	2.0125
3	1.2900	1.6707	2.0147	2.3311	2.6263	3.1687	3.6630
4	1.6418	2.2971	2.9253	3.5313	4.1189	5.2494	6.3316
VSWR	1.0757	1.1265	1.1673	1.2025	1.2340	1.2901	1.3400
THETA	ATTENUATION OF L/D FILTER (DB)						
60	11.5307	15.2037	17.2283	18.6108	19.6560	21.1947	22.3199
75	18.9437	22.1369	23.8909	25.0960	26.0133	27.3751	28.3807
90	10.6134	11.9189	12.7114	13.3043	13.7868	14.5571	15.1682

TABLE VI

DOUBLE-ORDERED TRANSMISSION ZERO, $n = 2$, $\theta_1 = 15°$, $\theta_2 = 30°$

R	2	3	4	5	6	8	10
STUB							
1	1.1722	1.4483	1.6274	1.7632	1.8742	2.0520	2.1941
2	2.7630	3.5372	4.1414	4.6563	5.1129	5.9101	6.6024
3	3.6678	5.4268	7.0499	8.5900	10.0686	12.8851	15.5567
4	1.2070	1.9391	2.6802	3.4310	4.1901	5.7289	7.2896
LINE							
1	.7867	.8375	.8818	.9200	.9536	1.0111	1.0596
2	.9108	1.1233	1.3112	1.4805	1.6358	1.9154	2.1649
3	.7930	1.0609	1.3198	1.5690	1.8099	2.2711	2.7104
VSWR	1.0536	1.0890	1.1171	1.1411	1.1625	1.2002	1.2334
THETA	ATTENUATION OF L/D FILTER (DB)						
45	14.6560	18.6839	20.8601	22.3324	23.43933	25.0596	26.2383
60	28.7220	32.6040	34.6968	36.1154	37.1840	38.7524	39.8966
75	33.0610	36.6023	38.5385	39.8627	40.8665	42.3495	43.4384
90	24.5315	26.8156	28.1507	29.1078	29.8595	31.0124	31.8909

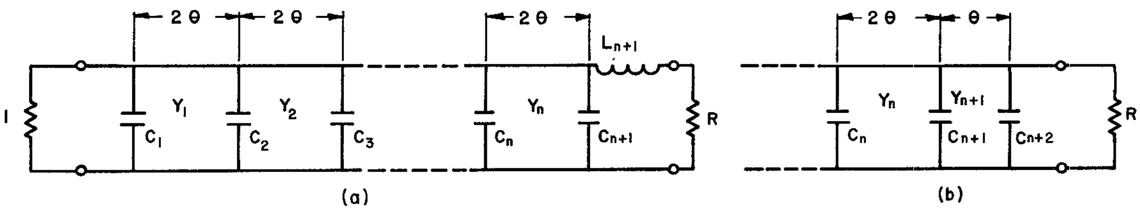

Fig. 4. (a) Basic double-ordered prototype circuit. (b) After addition of redundant TLE and Kuroda transformation. Lumped components shown are stubs of length θ in prototype.

it has restricted application because it is found that the lower impedance in the case of the shunt capacitive transformer is always at the end without the capacitor. It would not be possible to match into a low impedance with shunt parasitic capacitance, a common requirement in the design of varactor multipliers, for example.

The difficulty is overcome in one of two ways.

1) Introduce a redundant TLE at the end with no shunt stub, and transform part of the previous shunt stub across the two TLEs by means of Kuroda's identity [1, p. 765] to give a double-length TLE of uniform impedance. This procedure is illustrated in Fig. 3. The final version shown in Fig. 3(c) is the one given in Tables I–V.

2) Synthesize prototypes having second-ordered transmission zeros (or attenuation poles). Matrix (5) applies to the single-ordered case having $2n$ TLEs and one effective shunt stub. If this is multiplied by a series stub having transfer matrix

$$\begin{bmatrix} 1 & Zt \\ 0 & 1 \end{bmatrix} \tag{8}$$

the degree becomes

$$D\begin{bmatrix} 2n & 2n-1 \\ 2n+1 & 2n \end{bmatrix} \cdot D\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = D\begin{bmatrix} 2n & 2n+1 \\ 2n+1 & 2n+2 \end{bmatrix} \tag{9}$$

which is capable of satisfying condition (4). This prototype is shown in Fig. 4, with the termination of Fig. 4(a). Alternatively the series stub may be replaced by a redundant TLE of length θ and another shunt stub by means of Kuroda's identity, leading to the termination of Fig. 4(b). This is the form of network given in Table VI.

Note that the only disadvantage of the single-length TLE would be at high frequencies where it may become rather short. It does not affect the accuracy of conversion to a semilumped filter. Indeed prototypes having higher ordered transmission zeros could be synthesized with an arbitrary mixture of single and double TLEs in the cascade, but this would appear to offer no advantages for the simple terminations under consideration. They could be useful as matching networks where more complex parasitic situations at either or both terminations exist.

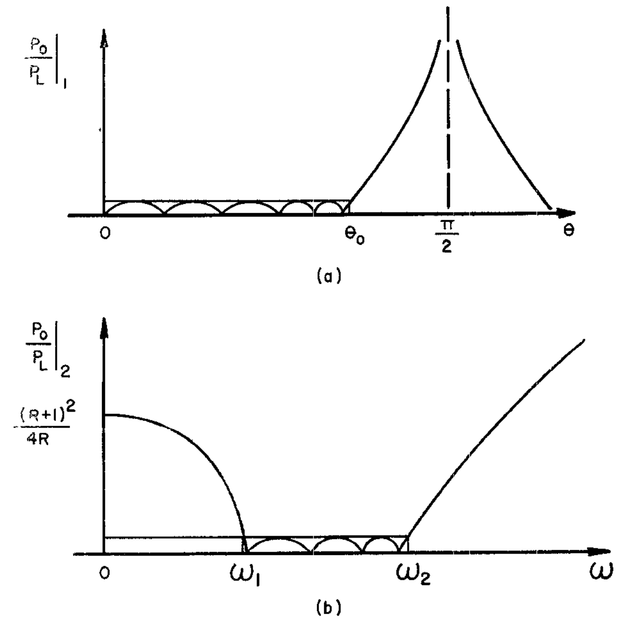

Fig. 5. (a) Insertion loss of commensurate distributed filter with mth-ordered pole. (b) Insertion loss of lumped element impedance-transforming filter.

Solution of the Approximation Problems

A. Single-Ordered Transmission Zero

The insertion loss is equiripple on the θ axis between θ_1 and θ_2, is equal to $(R+1)^2/4R$ at $\theta=0$, and has a single-ordered pole at $\theta=\pi/2$. It must lead to the derivation of a transfer matrix of degree given by (7). The function required is suggested by known solutions to two other different approximation problems, each having characteristics similar in part to that of Fig. 1. Combining the ideas of these leads to the solution in the present case.

The first of these is the solution to the problem of the general commensurate transmission-line filter consisting of a cascade of n TLEs and m nonredundant stubs [9], [10]. In this case the insertion loss function for the Chebyshev low-pass filter shown in Fig. 5(a) is

$$\left.\frac{P_0}{P_L}\right|_1 = 1 + h^2 \cos^2\left(n \cos^{-1}\frac{s}{s_0} + m \cos^{-1}\frac{\tau}{\tau_0}\right) \tag{10}$$

where

$$\frac{s}{s_0} = \frac{\sin\theta}{\sin\theta_0}, \qquad \frac{\tau}{\tau_0} = \frac{\tan\theta}{\tan\theta_0} \tag{11}$$

and in (10) and in similar functions derived later, the trigonometric cos becomes hyperbolic cosh when the arguments in the inverse functions become greater than unity.

The second case is the lumped element quasi-low-pass impedance-transforming filter referenced in [2], for which the Chebyshev equiripple characteristic is shown in Fig. 5(b). The insertion loss function for the case of degree $2n$ is

$$\left.\frac{P_0}{P_L}\right|_2 = 1 + h^2 T_n^2\left(\frac{2\omega^2 - \omega_2^2 - \omega_1^2}{\omega_2^2 - \omega_1^2}\right) \tag{12}$$

where

$$\frac{R-1}{2\sqrt{R}} = hT_n\left(\frac{\omega_2^2 + \omega_1^2}{\omega_2^2 - \omega_1^2}\right) \tag{13}$$

and T_n is the first-kind Chebyshev polynomial of degree n. In the case of a single-ordered transmission zero, $m=1$ in (10), in which case it may be written in the form

$$\left.\frac{P_0}{P_L}\right|_1 = 1 + h^2\left[\frac{(1+\cos\theta_0)T_{2n}\left(\dfrac{\sin\theta}{\sin\theta_0}\right) - (1-\cos\theta_0)T_{2n-1}\left(\dfrac{\sin\theta}{\sin\theta_0}\right)}{\cos\theta}\right]^2. \tag{14}$$

Equations (12)–(14) suggest that the solution to the insertion loss approximation problem of Fig. 1 for a cascade of $2n$ TLEs and a single nonredundant stub may be given as

$$\frac{P_0}{P_L} = 1 + h^2\left[\frac{\alpha T_n(s') - \beta T_{n-1}(s')}{\cos\theta}\right]^2 \tag{15}$$

where

$$s' = \frac{2s^2 - s_1^2 - s_2^2}{s_2^2 - s_1^2} \tag{16}$$

$$s = \sin\theta, \qquad s_1 = \sin\theta_1, \qquad s_2 = \sin\theta_2. \tag{17}$$

Here the choice of the arguments of the Chebyshev functions to be s' as in (16) ensures that the equiripple range is completely traversed in the interval $\theta_1 < \theta < \theta_2$. The parameters h, α, and β of (15) may now be chosen to give the correct values at $\theta=0$, $\theta=\theta_1$, and $\theta=\theta_2$, leading to the values

$$\alpha = \frac{\cos\theta_2 + \cos\theta_1}{2} \qquad \beta = \frac{\cos\theta_1 - \cos\theta_2}{2} \tag{18}$$

$$\frac{R-1}{2\sqrt{R}} = h[\alpha T_n(s_0') + \beta T_{n-1}(s_0')] \tag{19}$$

where

$$s_0' = \frac{s_2^2 + s_1^2}{s_2^2 - s_1^2}. \tag{20}$$

It is simple to show that (15) is an equiripple function.

B. Double-Ordered Transmission Zero

Here the more general case of a $2m$th-ordered transmission zero will be derived. Equations (10)–(12) suggest that in this case the insertion loss may be given as

$$\frac{P_0}{P_L} = 1 + h^2\cos^2(n\cos^{-1}s' + m\cos^{-1}\tau') \tag{21}$$

where s', s, s_1, and s_2 are defined as in (16)–(17), and

$$\tau = \tan\theta \tag{22}$$

with τ', τ_1, and τ_2 being defined from (16) and (17) by replacing s everywhere by τ.

The proof that (21) is a rational function of the desired form for the problem under consideration is given in the Appendix. The case of interest here for the circuit shown in Fig. 4 has $m=1$. It is shown in the Appendix that in this case (21) can be written in the following form, which is more convenient for synthesis:

$$\frac{P_0}{P_L} = 1 + \left[\frac{hP_{2n+2}(s)}{c^2}\right]^2 \tag{23}$$

where

$$s = \sin\theta, \qquad c = \cos\theta \tag{24}$$

and $P_{2n+2}(s)$ is a polynomial of degree $2n+2$ in $\sin\theta$ given by

$$P_{2n+2}(s) = \tfrac{1}{4}(c_1+c_2)^2T_{n+1}(s') + \tfrac{1}{4}(c_1-c_2)^2T_{n-1}(s') + \tfrac{1}{2}[c_2^2 - c_1^2]T_n(s') \tag{25}$$

where s' is given by (16) and the subscripts to s and c defined in (24) refer to the angles θ_1 and θ_2 of Fig. 1. The impedance-transformation ratio R is determined by the value of P_0/P_L at $\theta=0$, i.e.,

$$\frac{R-1}{2\sqrt{R}} = P_{2n+2}(0) \tag{26}$$

which, as in the first-ordered case, determines the value of the passband ripples by fixing the value of h in (23).

The Synthesis

The synthesis of either type A or type B insertion loss functions in the form shown in Figs. 3 and 4 is fairly standard. It must be performed by computer since the calculations are lengthy, and a high degree of accuracy must be maintained throughout. Details of the synthesis may be found in previous papers, e.g., [5], [8].

There is no guarantee that the element values derived by the synthesis will be all positive. However, the only cases so far computed where one or two shunt stubs became negative occurred for $\theta_2 > \pi/4$, and the circuits appear to be realizable for all useful values of the parameters.

The Tables

Tables I–V give representative examples of prototypes with single-ordered transmission zeros, and Table VI presents double-ordered cases. The transforming and filtering properties of the prototypes are very good for bandwidths to 3:1 and impedance ratios to 10:1 with $n \leq 4$, and these are the limiting values chosen for tabulation. Tables I–VI give values corresponding to the following parameters:

n	2, 3, 4
Band edges θ_1, θ_2	15, 30 and 15, 45
Immitance ratio R	2, 3, 4, 5, 6, 8, 10

Tables I–V are for prototypes with single-ordered transmission zeros as illustrated in Fig. 3(c). The lowest degree case would be for $n=1$, but this does not give good values of VSWR for octave bandwidths, and is not tabulated here. The case $n=2$ for 3:1 bandwidth is not tabulated for the same reason.

Each column corresponds to the value of R denoted at the head. The first $(n+1)$ numbers are the normalized admittances of the open-circuited stubs of length θ. The next n numbers are the normalized admittances of the TLEs of length 2θ. The terminating admittances of the filter transformer are given as 1 and R, with the termination of admittance R following stub $(n+1)$. (Note that elsewhere in the text R may have the dimensions of impedance, e.g., (3), (4). The next row gives the peak value of the VSWR in the passband. Finally the filter properties are illustrated by giving the attenuation of the derived mixed lumped and distributed filter at values of θ (defined in Fig. 1) from $\theta_2+15°$ through 90° in increments of 15°. The attenuation has a rather sharp minimum just above 90°, and then increases again. The derived mixed lumped and distributed filter is given by replacing all shunt stubs $y_1, y_2, \cdots, y_{n+1}$ by lumped capacitors $C_1, C_2, \cdots, C_{n+1}$ through application of (1). This process is illustrated in the next section.

Table VI presents cases with doubled-ordered transmission zeros for $n=2$, $\theta_1=15°$, $\theta_2=30°$. The general circuit is shown in Fig. 4(b), which indicates that the first n lines are of length 2θ, but the final TLE is of length θ. Note that the double-ordered case gives one more TLE and one more stub than the corresponding value of n for the single-ordered case. Indeed the values given in Table VI are similar to those given in Table II. Only one table is presented for the double-ordered cases since the single-ordered cases are expected to find dominant use.

The computer program written in Basic occupies less than 8K of core, and the total computation time for the complete set of tables was approximately 25 seconds, using a CDC 3600 time-sharing computer service.

A number of interesting and useful features mentioned earlier may be noted by inspection. In all cases the line admittances are less than the terminating admittance R, often considerably less. In many cases these admittances also are all greater than unity, so that the admittance level of the transformer lies within

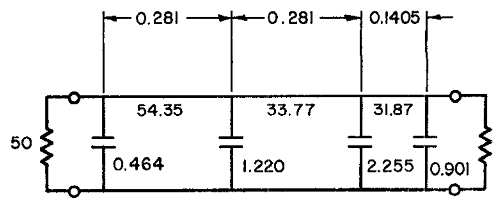

Fig. 6. Mixed lumped and distributed 50–10-Ω transformer for 3.5–7.0 GHz. Impedances in ohms, capacitances in picofarads, line lengths in inches.

the range of admittances encompassed by the two terminating values for such cases. As n increases the admittance level becomes more oscillatory, but the VSWR is already rather good to excellent at $n=4$, with a useful stopband attenuation. When the prototype is converted to the L/D filter the VSWR deteriorates slightly, so that for this and practical reasons there is no point in choosing a prototype VSWR which is exceptionally close to unity.

Example

This example is for a double-ordered case, since this was the first to be synthesized. The technique would be almost identical to the case of the single-ordered transmission zero, which as stated would usually be preferred. The example may be stated in problem form as follows.

Design a filter for the passband 4.0–6.5 GHz to transform from 50 Ω to 10 Ω having passband VSWR better than 1.2 and attenuation >20 dB in the band 12–19.5 GHz.

The passband is centered in the octave band 3.5–7.0 GHz, and if we design for this we should have 0.5 GHz of excess bandwidth on each side, which gives a useful safety factor (normally we would not want to use fine tuning). The prototype chosen here is for $n=2$, $R=5$, given in Table VI, with a VSWR of 1.1411. The attenuation is greater than 20 dB, at least from $\theta=45°$ through 90°, corresponding to the band 10.5–21 GHz (since $\theta=30°$ at 7.0 GHz).

The normalized susceptances of the lumped shunt capacitors at $\theta=30°$ are calculated from (1), noting that we are now normalized to Z Ω rather than 1 Ω, i.e.,

$$\omega_0 Z C_1 = 1.7632 \tan 30 = 1.0180$$

and similarly

$$\omega_0 Z C_2 = 2.6883 \quad \omega_0 Z C_3 = 4.9594 \quad \omega_0 Z C_4 = 1.9809.$$

Setting $\omega_0 = 2\pi \times 7 \times 10^9$ rad/s and $Z=50$ Ω, the shunt capacitors are, in picofarads,

$$C_1 = 0.464 \quad C_2 = 1.220 \quad C_3 = 2.255 \quad C_4 = 0.901.$$

The mixed lumped and distributed filter design is shown in Fig. 6, and its theoretical performance compared with that of the distributed prototype in Fig. 7. Where no dotted line is shown the difference between the mixed lumped and distributed filter and the prototype is negligible e.g., this is so for the passband VSWR.

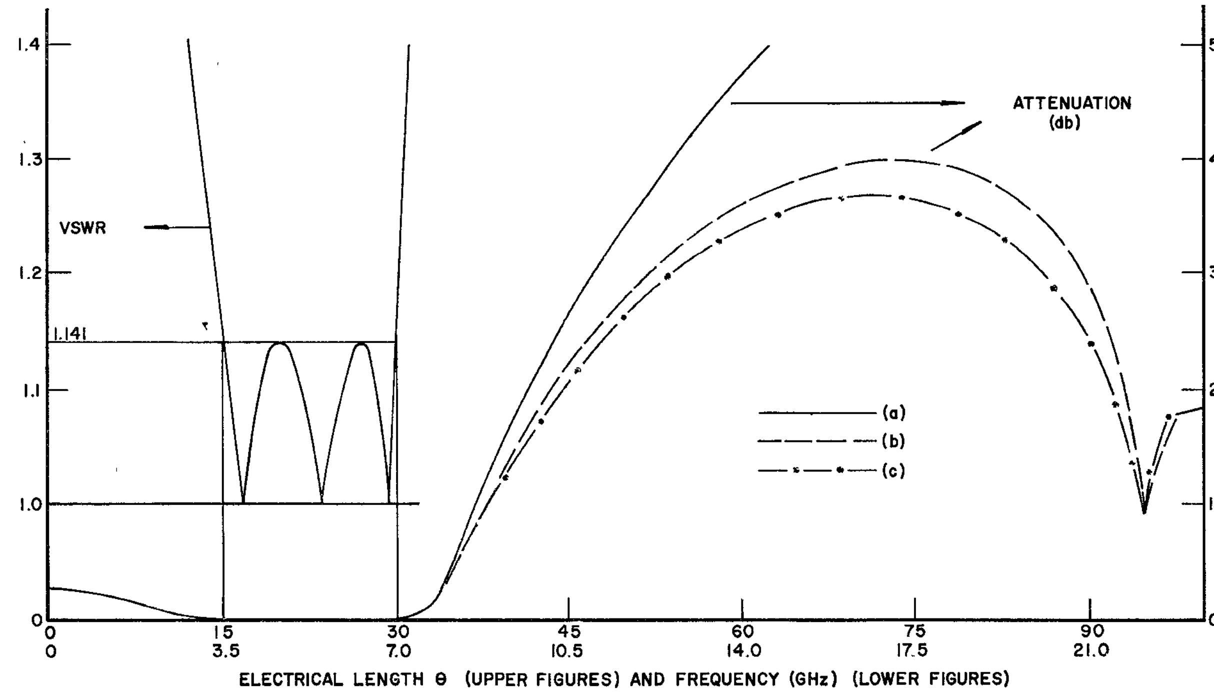

Fig. 7. Comparison of theoretical characteristics. (a) Prototype transformer. (b) Mixed lumped and distributed transformer of Fig. 5. (c) Coaxial realization of Fig. 7.

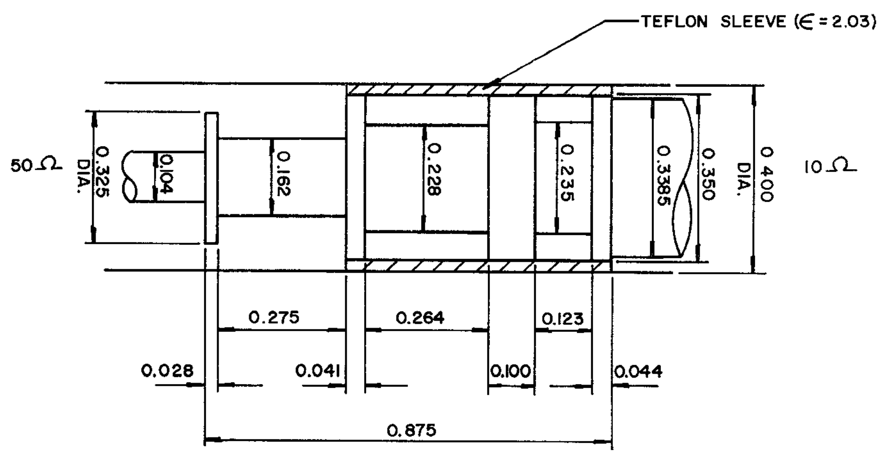

Fig. 8. 50–10-Ω impedance-transforming filter realized in coaxial line.

This ideal mixed lumped and distributed filter transformer is 0.703 in long in air, compared to 1.26 in for the conventional three-section quarter-wave transformer required for an octave bandwidth. The latter would require an additional low-pass filter in cascade to meet the stopband attenuation requirement.

A Practical Realization and Experimental Results

The mixed lumped and distributed filter of Fig. 6 may be realized in a variety of media, e.g., in strip transmission line, microstrip, or coaxial line. The latter was chosen for convenience, and a dimensioned diagram of the device is given in Fig. 8. It is slightly longer than the ideal mixed lumped and distributed transformer of Fig. 6 because the disk capacitors are quite thick. The computed performance given in Fig. 7 also shows a slightly lower level of stopband attenuation. In the design of Fig. 8 due account has been taken of fringing capacitances and mutual interactions [12]. The line lengths between the disk capacitors were adjusted to allow for the extra phase shift due to the finite disk thickness. The equivalence was carried out at the cutoff frequency of 7 GHz, which in theory should be realized exactly.

Two transformers of this type were manufactured and connected together back to back, spaced by a 0.420-in length of 10-Ω line. The comparison between the computed and measured performance of this double transformer is given in Fig. 9. The two transformers

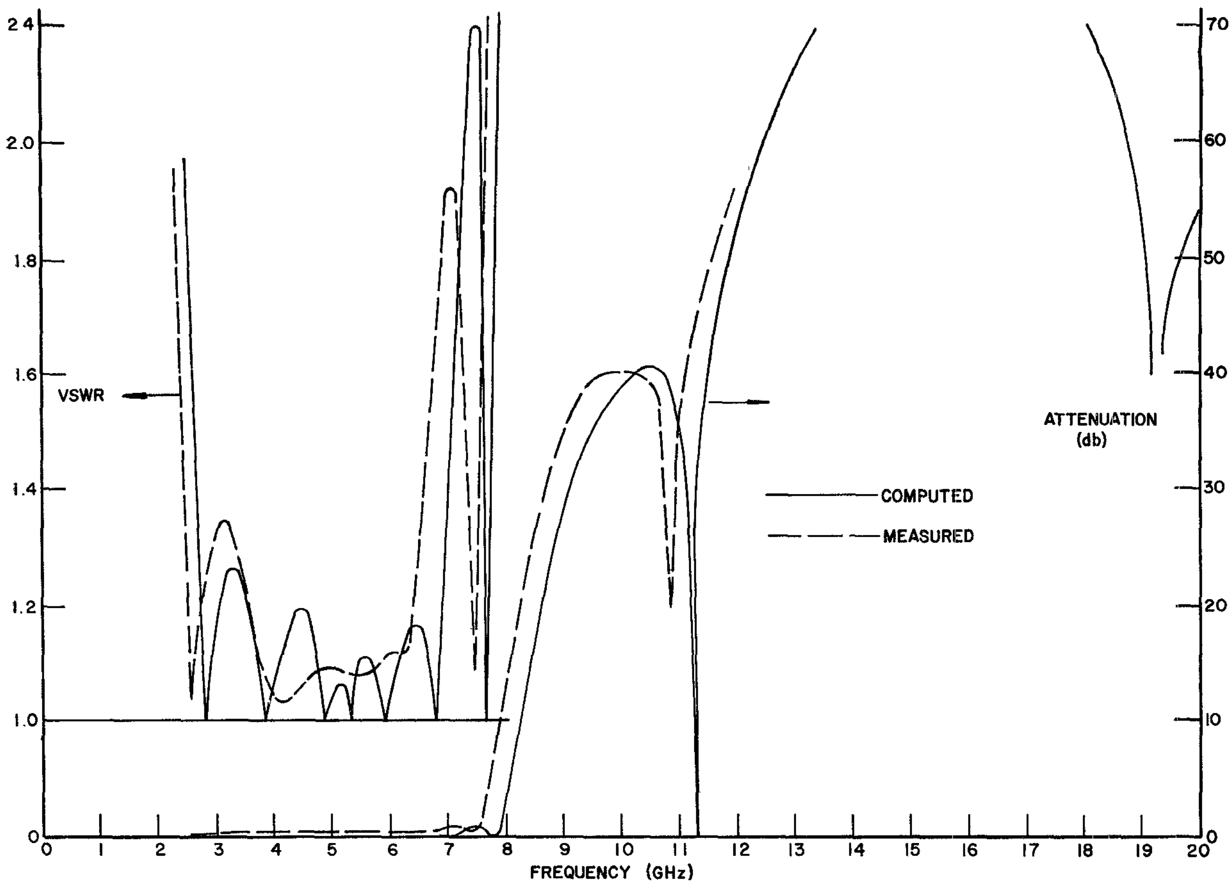

Fig. 9. Computed and measured performance of two back-to-back transformers (Fig. 7) spaced by 0.42 in of 10-Ω line.

interact considerably to give a very different presentation to that of the individual transformer whose characteristics are given in Fig. 7. It is noteworthy how closely the large ripples in both passband and stopband caused by the interaction are reproduced in the experiment. The only significant deviation from theory is a downward shift of the entire characteristic by an amount of approximately 0.4 GHz. A likely cause of this is that the fringing capacitances of the disks were calculated assuming that the dielectric has no effect, whereas in practice they are almost certainly increased. The experimental results show that apart from the frequency shift the performance of an individual transformer is substantially realized as the theoretical characteristic of Fig. 7.

Note on Broad-Band Matching

The new class of mixed lumped and distributed transformers may be used to match between two impedances having simple parasitics, such as shunt capacitances or series inductances, or occasionally when one impedance has an *RLC* equivalent circuit [see Fig. 4(a)]. As stated previously, more complicated situations could be handled by synthesizing an appropriate type of filter. The main point to note here is that the filter transformers tabulated in this paper are not optimum for the broad-band matching case. Here the optimum must give a passband reflection coefficient which is as close as possible to a constant, and certainly no reflection coefficient zeros are allowable [13]–[15]. The optimum cases are derived by adding a positive parameter to the insertion loss function of (15) or (23). This has the effect of preventing the occurrence of reflection coefficient zeros in the passband. In any given matching situation the value of the parameter is chosen to give a minimum mismatch over the desired passband, as described in [13]–[15].

However, in many practical situations the synthesis of specially optimized matching functions may not be justified, and quite acceptable results may be obtained by direct application of the circuits tabulated in this paper.

Conclusions

The new type of impedance transformer may be safely specified for many applications where multistep quarter-wave transformers are now used, giving a considerable saving in length. The technique has not yet been tested for very low VSWR applications (<1.05) and further research would be necessary here. However, it is safe to predict that this could certainly be obtained by tuning.

The harmonic filtering properties of the new transformer will be of advantage in many instances, enabling both a conventional transformer and a low-pass filter to be replaced. It may be realized in any type of transmission line (including waveguides).

The transformer realization has been presented as a cascade of lumped element shunt capacitors and transmission lines, avoiding the introduction of inductors. Possibly the dual network consisting of series inductors and transmission lines may be of use in some rare applications.

The impedance-transforming filters may also be used as convenient broad-band matching networks. Here for optimum results or for more complicated matching problems an appropriate new transfer function would be specified, but this would then require synthesis by computer.

The facility of the distributed prototype network approach to the design of mixed lumped and distributed two-port networks may be considered clearly demonstrated in this and a previous paper [8]. It would be outmoded by the appearance of a solution to the two-variable approximation problem associated with a synthesis technique, which would need to be simple to apply in practical situations.

Appendix

Proof that Insertion Loss Function (21) is Rational

The rational Chebyshev polynomials are defined as follows:

First kind:

$$T_n(x) = \begin{cases} \cos n \cos^{-1} x, & |x| \leq 1 \\ \cosh n \cosh^{-1} x, & |x| \geq 1. \end{cases} \tag{27}$$

Second kind:

$$U_n(x) = \begin{cases} \dfrac{\sin n \cos^{-1} x}{\sin \cos^{-1} x}, & |x| \leq 1 \\ \dfrac{\sinh n \cosh^{-1} x}{\sinh \cosh^{-1} x}, & |x| \geq 1. \end{cases} \tag{28}$$

Note that $\sin \cos^{-1} x = \sqrt{1-x^2}$, etc.

Using the abbreviation of the type introduced in (16),

$$s' = \frac{2s^2 - s_1^2 - s_2^2}{s_1^2 - s_2^2} \qquad t' = \frac{2t^2 - t_1^2 - t_2^2}{t_1^2 - t_2^2} \tag{29}$$

then (21) becomes

$$\frac{P_0}{P_L} = 1 + h^2 F^2(s, t) \tag{30}$$

where

$$F(s, t) = \cos (n \cos^{-1} s' + m \cos^{-1} t'). \tag{31}$$

This may be expanded to give

$$F(s, t) = T_n(s')T_m(t') - U_n(s')U_m(t')\sqrt{(1 - s'^2)(1 - t'^2)}. \tag{32}$$

Hence $F(s, t)$ will be a rational function in s and t if $\sqrt{(1-s'^2)(1-t^1)^2}$ is rational. It is simple to show that

$$\sqrt{(1 - s'^2)(1 - t'^2)} = \frac{c_1c_2}{c^2}(1 - s'^2) \tag{33}$$

which is indeed rational, and this completes the proof.

In the case where $m = 1$, (32) becomes

$$F(s, t) = t'T_n(s') - \frac{c_1c_2}{c^2} U_n(s')(1 - s'^2). \tag{34}$$

Another relationship between t' and s' is that

$$t' = \frac{(2 - s_1^2 - s_2^2)s' + (s_1^2 - s_2^2)}{2c^2}. \tag{35}$$

Using also the following well-known (and simply derived) properties of the Chebyshev polynomials,

$$T_{n+1}(s') + T_{n-1}(s') = 2s'T_n(s') \tag{36}$$

$$T_{n+1}(s') - T_{n-1}(s') = -2(1 - s'^2)U_n(s') \tag{37}$$

then (34) becomes

$$\begin{aligned} F(s, t) &= \{[1 - \tfrac{1}{2}(s_1^2 + s_2^2) + c_1c_2]T_{n+1}(s') + [1 - \tfrac{1}{2}(s_1^2 + s_2^2) - c_1c_2]T_{n-1}(s') + (s_1^2 - s_2^2)T_n(s')\}/2c^2 \\ &= \{\tfrac{1}{2}(c_1 + c_2)^2 T_{n+1}(s') + \tfrac{1}{2}(c_1 - c_2)^2 T_{n-1}(s') + (c_2^2 - c_1^2)T_n(s')\}/2c^2. \end{aligned} \tag{38}$$

Substitution of (38) into (30) gives the final expression presented in (23)–(25).

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Short Papers

Measurement of Dielectric Materials Using a Cutoff Circular-Waveguide Cavity

J. HANFLING AND L. BOTTE

Abstract—**A technique is presented for accurately determining the dielectric constant of microwave materials. The concept is to resonate a cutoff circular-waveguide cavity by inserting the dielectric-disk sample. Unlike most dielectric measurement techniques which rely on perturbation methods, this one determines the dielectric constant from the absolute measurement of the resonant frequency. Also, the use of a cutoff cavity prevents false dielectric constant readngs by eliminating spurious resonances.**

I. Introduction

A cutoff circular-waveguide cavity is used for accurate and convenient measurement of the dielectric constant of microwave materials. The technique is applicable to all materials which can be formed into a circular disk.

The concept consists of locating a dielectric disk transversely at the center of a short-circuited circular-waveguide cavity. The dominant resonance of the cavity is for the TE_{11} mode even though the unfilled portion of the cavity is below cutoff. The dielectric constant of the disk material is determined from the resonant frequency. In practice the cavity is not short-circuited, but is weakly coupled to rectangular waveguide by coupling holes, as shown in Fig. 1(a). The features of the technique are the following: there are no higher mode resonances, the end effects introduced by coupling into and out of the cavity are accountable, and the samples are large providing good accuracy and reliability in determining the dielectric constant.

Using the parameters in Fig. 1(b), the formulas for determining the dielectric constant will be derived. Then the measured results and accuracies will be described.

II. Derivation of Formulas

In order to determine the dielectric constant of a disk, a relation between the measured cavity resonant frequency f_0 and the dielectric constant K of the disk is established. The desired formula is obtained by means of the "transverse-resonance" procedure [1], [2]. This procedure is valid since the generator and load impedances are loosely coupled to the disk; therefore, only reactive portions of the cavity need be considered. Referring to Fig. 1(b), the transverse-resonance condition is

$$\overleftarrow{Y} + \overrightarrow{Y} = 0. \tag{1}$$

When the center of the cavity is an open-circuit (OC) bisection, then the right-hand term in (1) becomes

$$\overrightarrow{Y} = jY_k \tan \theta_k \tag{2}$$

and the left-hand term in (1) is

$$\overleftarrow{Y} = -jY_a \cot \theta_a' \tag{3}$$

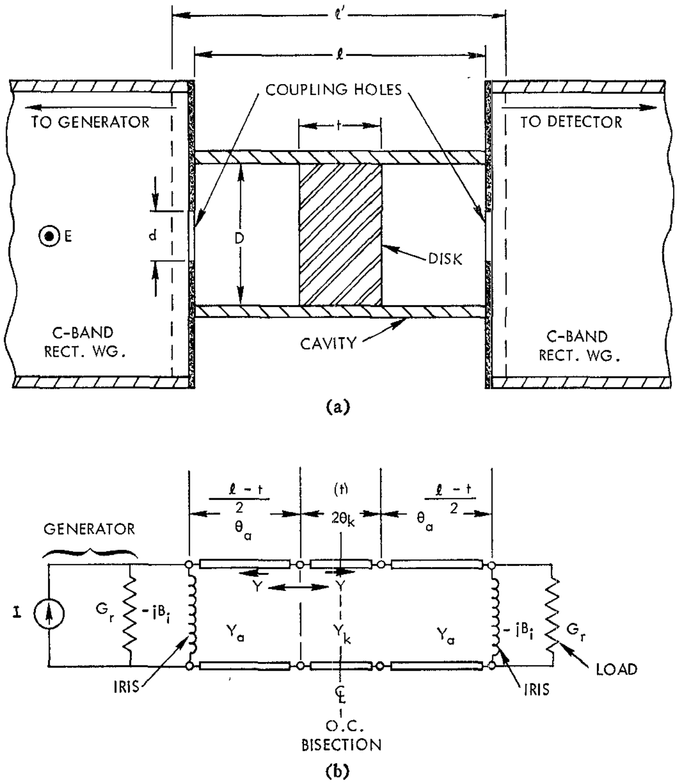

Fig. 1. (a) Cavity configuration. (b) Equivalent circuit of cavity.

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